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Application of Multi Layer Monte Carlo in Solving Partial Differential Equations for Bond Options Pricing

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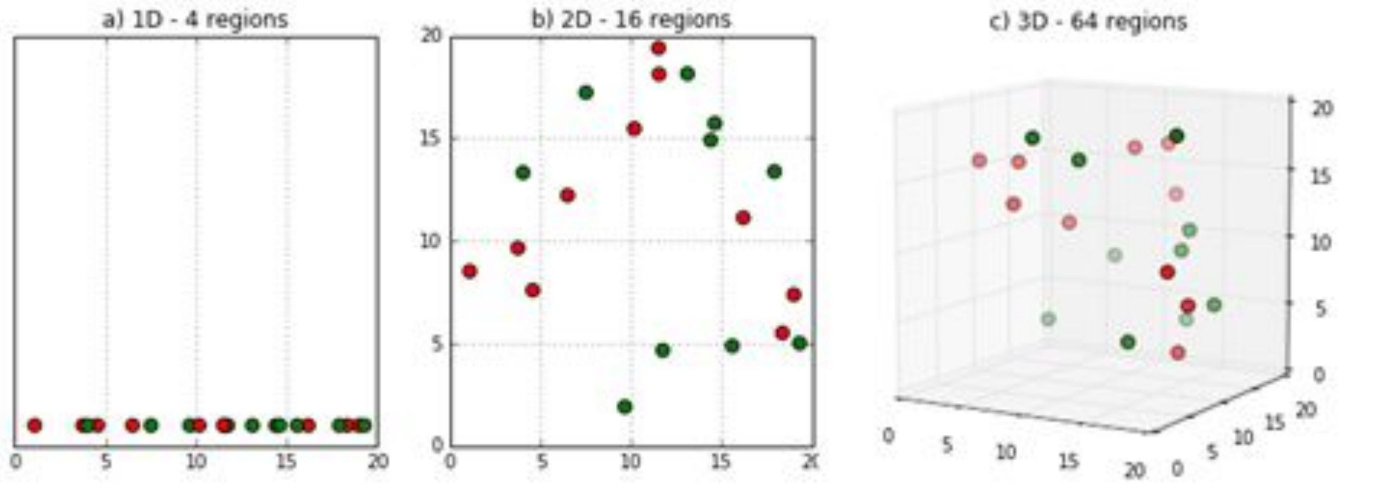
- Curse of Dimensionality

Motivation: Curse of Dimensionality (CoD)

Partial Differential Equations (PDEs): Black Scholes Barenblatt (BSB)

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

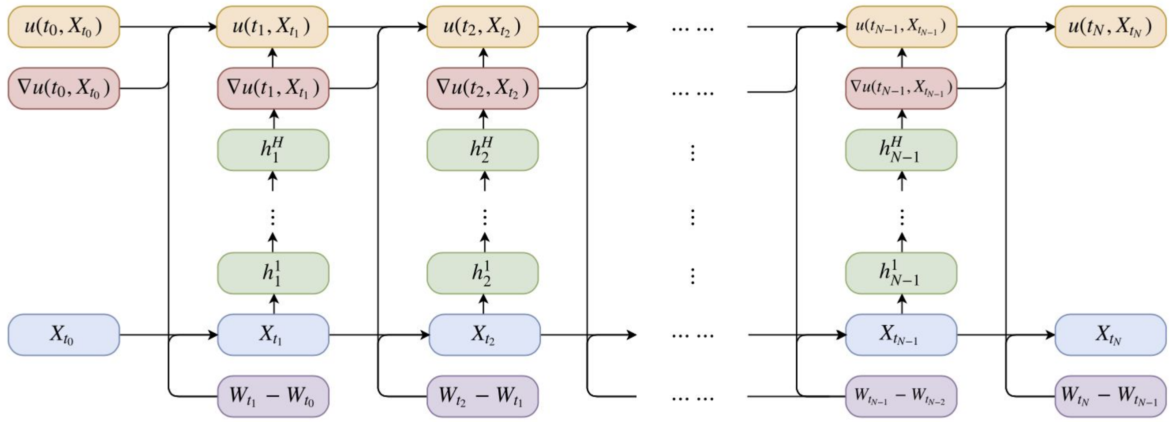
Curse of Dimensionality: More data required due to greater sparsity at larger dimensions



DeepAI, *Data Sparsity for Data Points*. .

Motivation: Time Efficiency and PDE Simplicity

<u>Deep Neural Network</u>: Solution for BSB CoD



<u>Limitation:</u> Time Inefficiency from Monte Carlo Sampling

<u>Limitation:</u> Model Simplicity

Fixed Interest Rate

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

<u>Goal:</u>

1) Increase Path Simulation Time Efficiency
2) Extend PDE Complexity for more realistic real world application

Weinan E, Jiequn Han, and Arnulf Jentzen. "Deep Learning-Based Numerical Methods for High-Dimensional Parabolic Partial Differential Equations and Backward Stochastic Differential Equations". In: Communications in Mathematics and Statistics 5.4 (Nov. 2017), pp. 349–380. issn: 2194-671X.doi:10.1007/s40304- 017- 0117- 6 .url: http://dx.doi.org/10.1007/s40304-017-0117-6

Background

- Forward Backward Stochastic Neural Networks (FBSNN) for High Dimensional PDEs
- Multilayer Monte Carlo (MLMC) Path Simulation

FBSNN: Initial Formulations

Quasi-linear PDEs

$$u_t = f(t, x, u, Du, D^2u)$$

Unknown solution: $u(t, x)$

Terminal Condition: $u(T, x) = g(x)$

Initial Coupled Forward Backward Stochastic Differential Equations (FBSDE) of general form:

$$\begin{aligned} dX_t &= \mu(t, X_t, Y_t, Z_t)dt + \sigma(t, X_t, Y_t)dW_t, \quad t \in [0, T], \\ X_0 &= \xi, \\ dY_t &= \varphi(t, X_t, Y_t, Z_t)dt + Z_t'\sigma(t, X_t, Y_t)dW_t, \quad t \in [0, T), \\ Y_T &= g(X_T), \end{aligned}$$

Solutions consist of stochastic processes X_T, Y_T, Z_T

Ito's Formula

$$Y_t = u(t, X_t), \text{ and } Z_t = Du(t, X_t).$$

Physics Informed Deep Learning

Automatic Differentiation

Approximate with a DNN $u(t, x)$

Maziar Raissi. Forward-Backward Stochastic Neural Networks: Deep Learning of High-dimensional Partial Differential Equations. 2018. arXiv:1804.07010 [stat.ML].

FBSNN: Modelling with DNN

Goal: Approximate *u(x,t)* with DNN

1. Discretize FBSDE with Euler-Maruyama scheme

$$X^{n+1} \approx X^n + \mu(t^n, X^n, Y^n, Z^n)\Delta t^n + \sigma(t^n, X^n, Y^n)\Delta W^n,$$
$$Y^{n+1} \approx Y^n + \varphi(t^n, X^n, Y^n, Z^n)\Delta t^n + (Z^n)'\sigma(t^n, X^n, Y^n)\Delta W^n,$$

2. Define Loss Function

$$\sum_{m=1}^{M}\sum_{n=0}^{N-1}|Y_m^{n+1} - Y_m^n - \Phi_m^n \Delta t^n - (Z_m^n)'\Sigma_m^n \Delta W_m^n|^2 + \sum_{m=1}^{M}|Y_m^N - g(X_m^N)|^2,$$

N = Number of Timesteps   M = Number of Path Simulations

FBSNN: Financial Applications

FBSD

$$dX_t = \sigma \mathrm{diag}(X_t) dW_t, \quad t \in [0, T],$$
$$X_0 = \xi,$$
$$dY_t = r(Y_t - Z_t' X_t)dt + \sigma Z_t' \mathrm{diag}(X_t) dW_t, \quad t \in [0, T),$$
$$Y_T = g(X_T), \;\; g(x) \;=\; \|x\|^2.$$

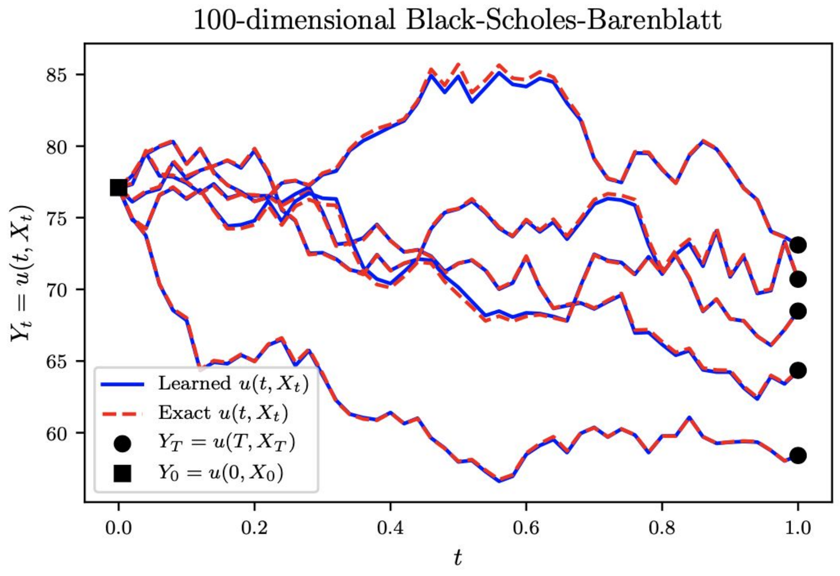
Black Scholes Barenblatt (BSB) PDE

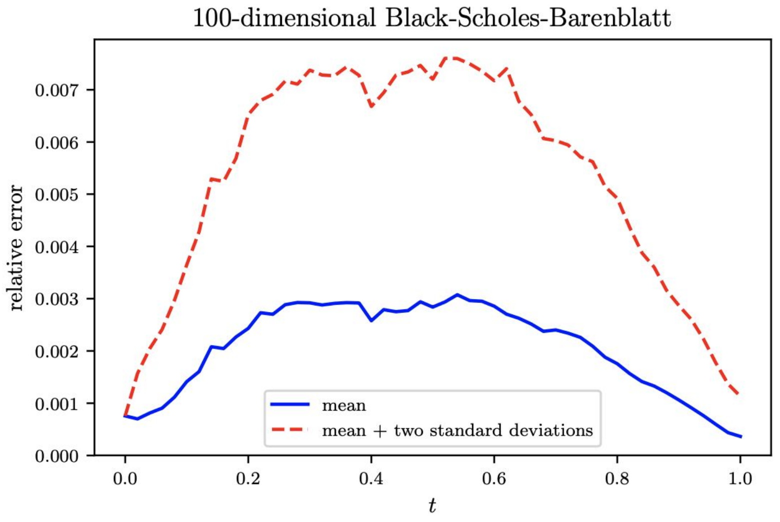
$$u_t = -\frac{1}{2}\mathrm{Tr}[\sigma^2 \mathrm{diag}(X_t^2) D^2 u] + r(u - (Du)'x)$$

Known Explicit Solution

$$u(t,x) = \exp\left((r+\sigma^2)(T-t)\right) g(x)$$

- X_t : Underlying Stock price at time t
- Y_t : Overlying derivative price at time t
- σ : Scalar Volatility
- r : Scalar Interest Rate
- **Monte Carlo Path Sampling** for data generation





FBSNN: Advancements and Limitations

Advancements

- Model parameters do not increase by number of timesteps (N)
- Converges to the exact value $Y_0 = u(0,X_0)$ in first few hundred iterations
- After 500 iterations, relative error of 2.3×10^{-3} is obtained

Limitations

- Monte Carlo sampling is costly
- More time steps required to more accurately estimate $Y_t = u(t,X_t)$ for $t > 0$
- Only basic PDEs modelled; BSB equations for calls and puts are more complex
- Only quasi-linear parabolic PDEs can be modelled and solved

MLMC: An Extension of Previous MC Method

- Previous MC method repeatedly iterates over a constant amount of points in time interval *[0,T]*
- MLMC introduces layers of different time steps covering different iterations

$0 < t < T$ $\quad 0 < l < L$ $\quad h_l = M^{-l}T$

$M = 2,\ T = 1$

$l = 1$, $h_l = 0.5$: $t_1 = 0.5$, N_1

$l = 2$, $h_l = 0.25$: $t_1 = 0.25$, $t_2 = 0.5$, $t_3 = 0.75$, N_2

$l = 3$, $h_l = 0.125$: N_3

...

T: Terminal Time
L: Terminal Layer
h_l: time step size at layer l
M: number of time step increase per layer
ε: relative error

Iterations at each level

$$N_l = \left\lceil 2\,\epsilon^{-2}\sqrt{V_l\,h_l}\,\left(\sum_{l=0}^{L}\sqrt{V_l/h_l}\right)\right\rceil$$

Convergence Condition $\left|\widehat{Y}_L - M^{-1}\widehat{Y}_{L-1}\right| < \frac{1}{\sqrt{2}}\,(M^2-1)\,\epsilon$

MLMC: Initial Paper Results

Theory

- To achieve an accuracy of $O(\varepsilon)$ in sampling , the computation cost is reduced from $O(\varepsilon^{-3})$ of MC to $O(\varepsilon^{-2}(\log\varepsilon)^2)$ of MLMC

Application on Option Pricing

- Time decrease of a factor in the range of 30-65 in comparison with MC or 4-10 in comparison to MC + performance boosting methods (ie: Richardson extrapolation)

Limitations

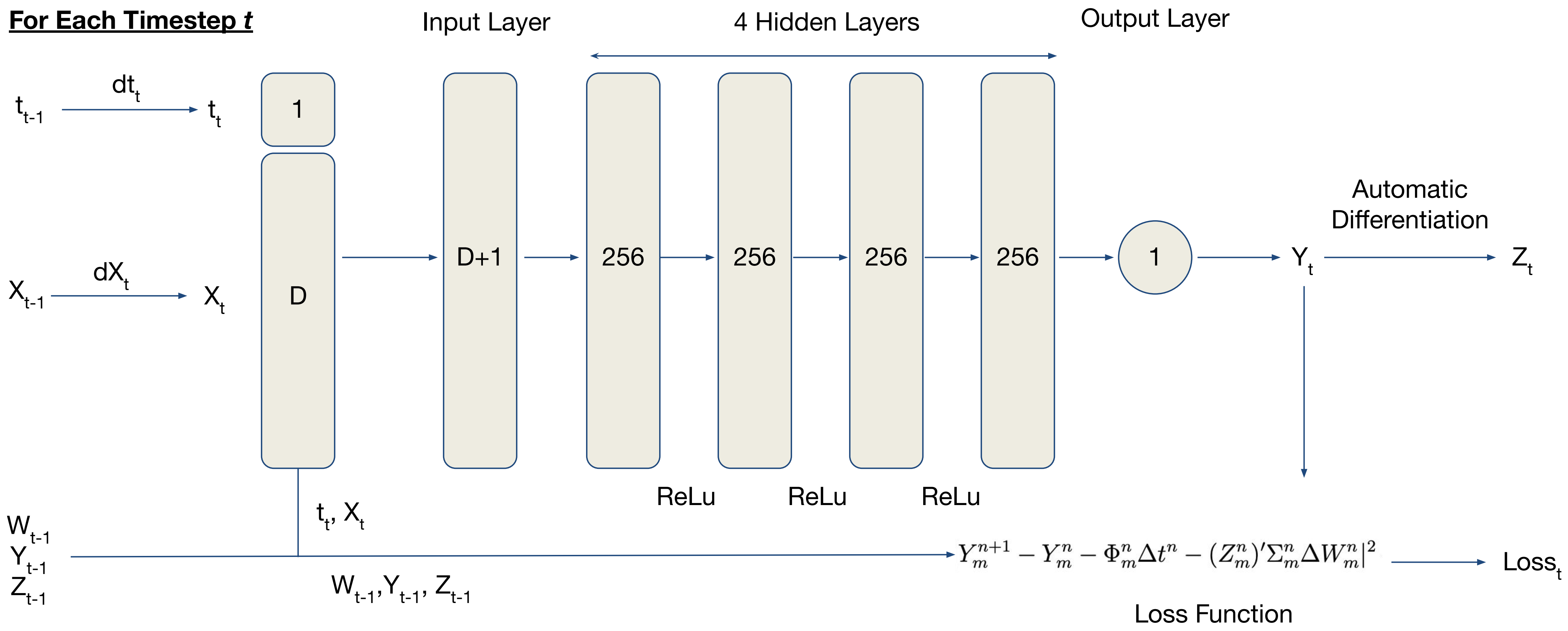
- Lack of complex finance applications (ie: high dimensionality) during testing
- Lack of significant improvement when combined with Richardson Extrapolation; other performance boosting methods (ie: Quasi Monte Carlo, Milstein Discretization) must be explored
- Unknown benefits when used for neural network data generation
- Potential problems arising due to bias from Monte Carlo path simulation is still present

Methods and Results

- Comparing MLMC and MC
- Exploring MLMC Hyperparameters
- Bond Options (Stochastic Interest Rates)

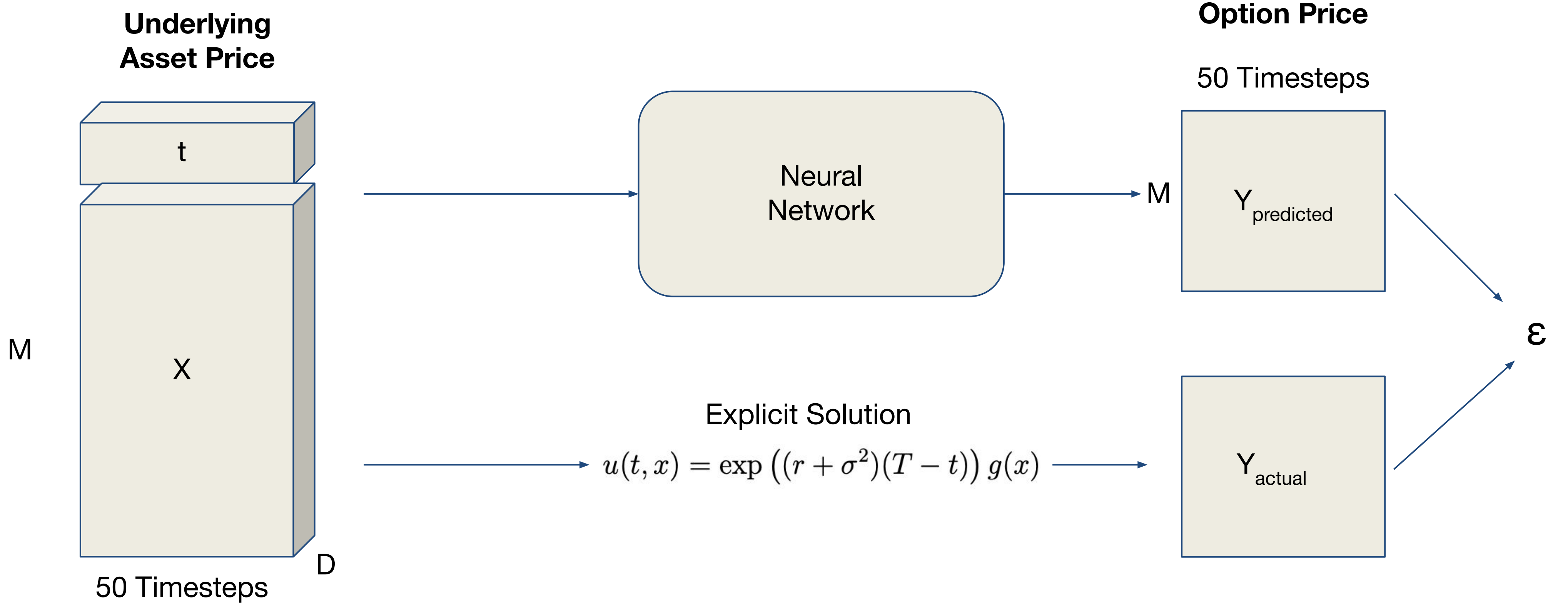
Methods: Comparing MLMC and MC (Training)

Performed across all M batches over N timesteps for each training iteration



Methods: Comparing MLMC and MC (Validation)

<u>For Each Iteration Until ε=User Specified Error</u>



Relative Error = $\varepsilon = |Y_{actual} - Y_{predicted}| / Y_{actual}$

Results: Comparing MLMC and MC

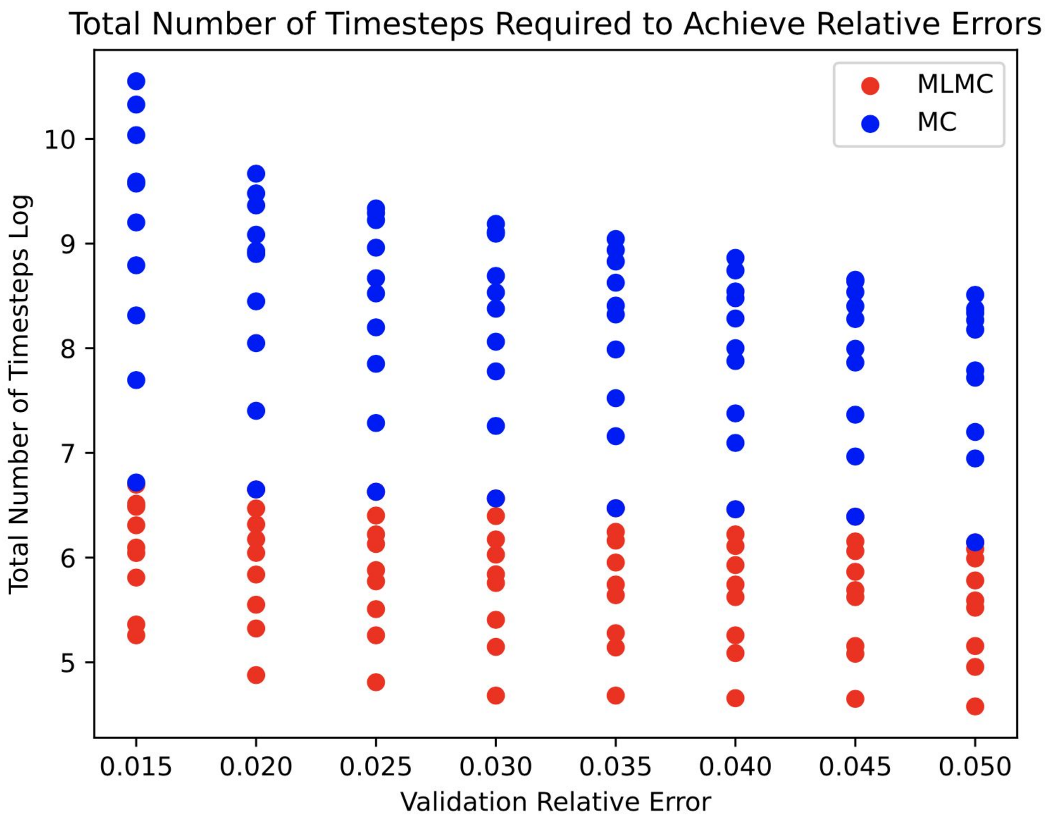


Figure 1
Best MC model requires 5.1-7.2x more timesteps than best MLMC model

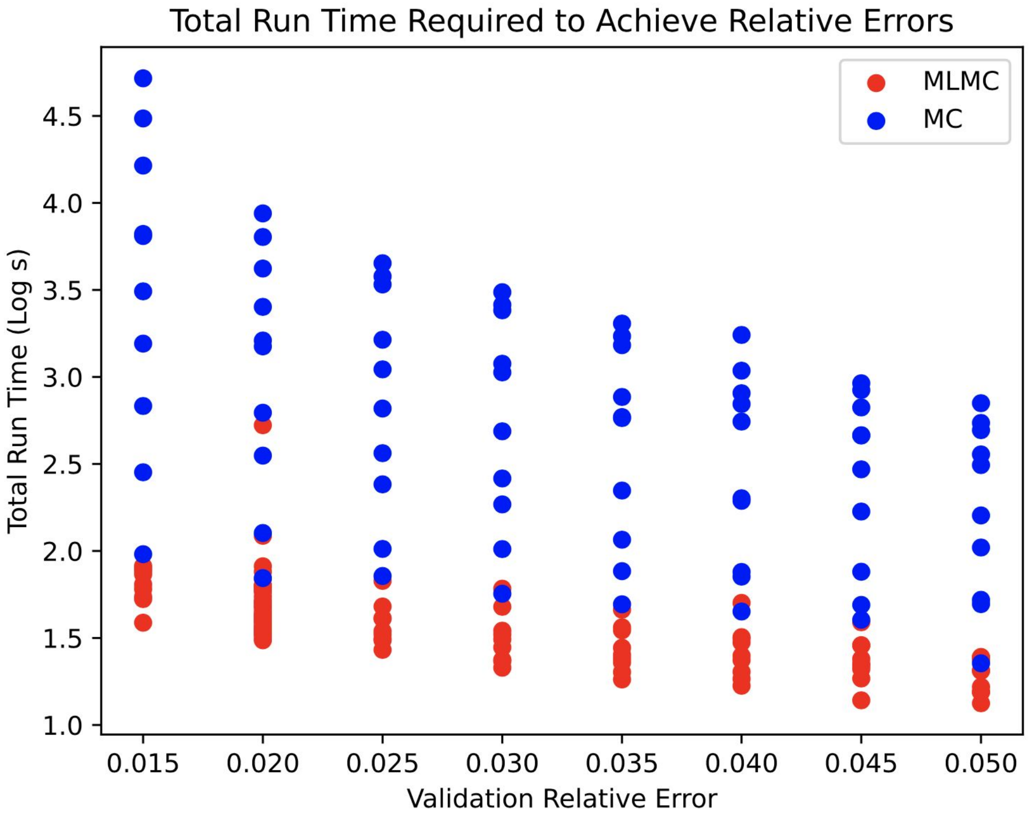
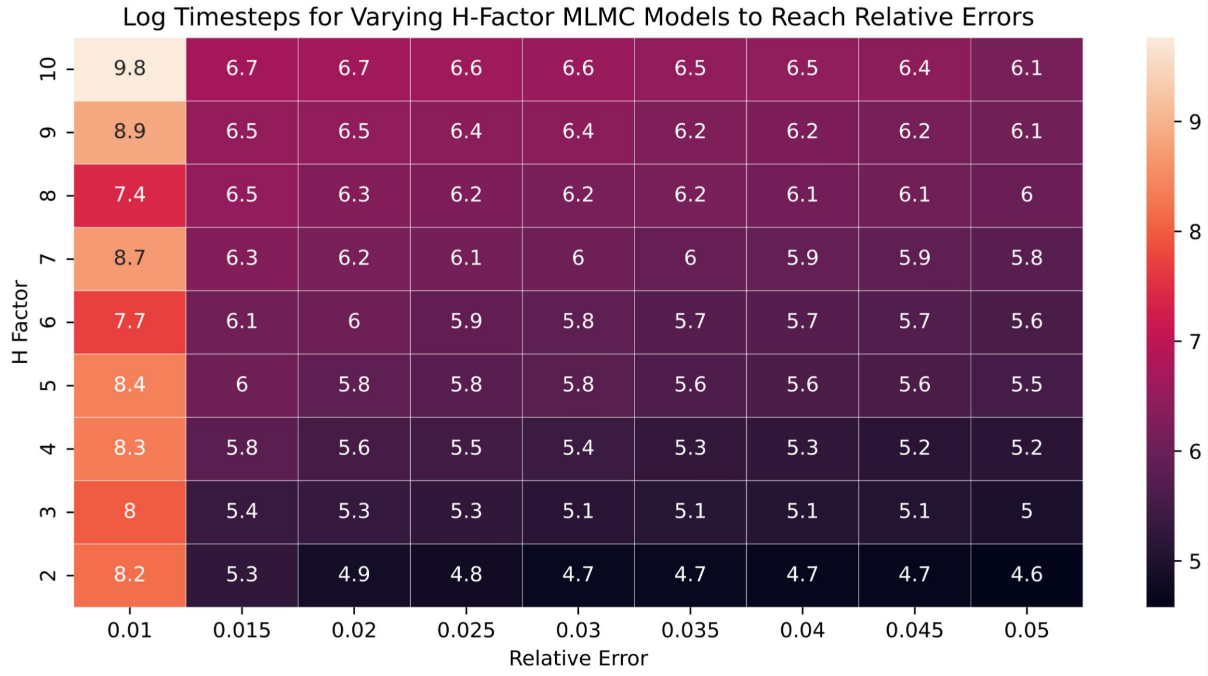
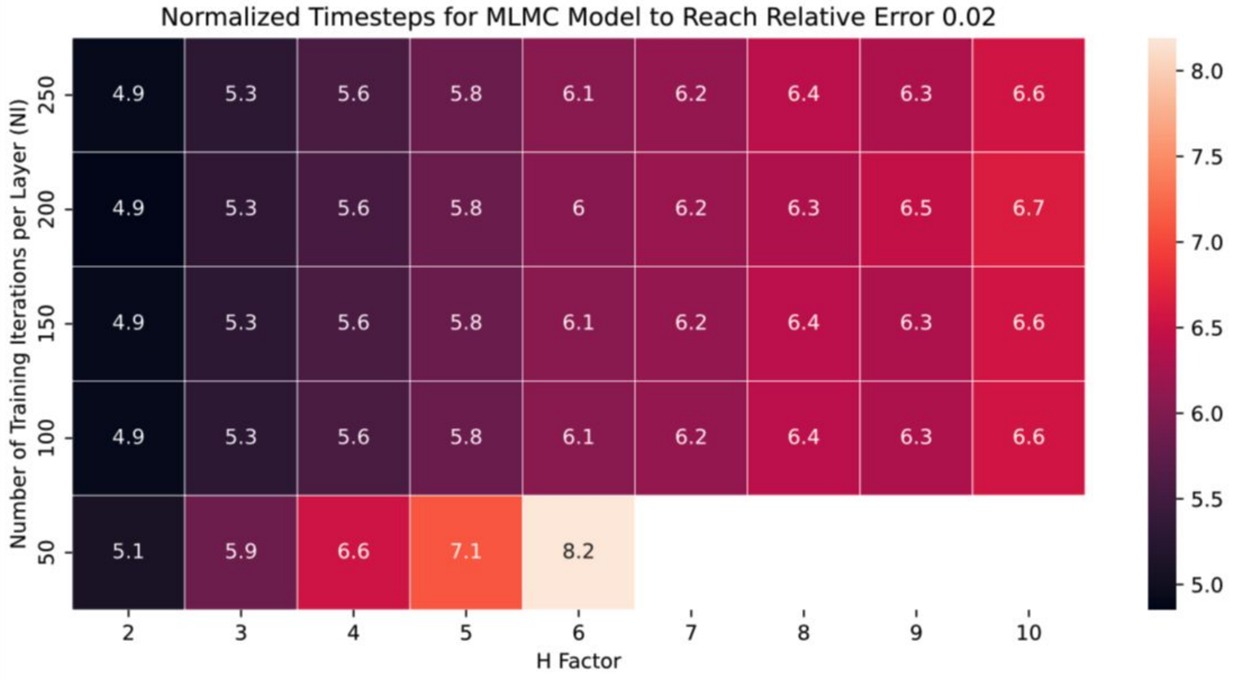


Figure 2
Best MC model requires 1.4-1.6x more time than the best MLMC model

Results: MLMC at Different Hyperparameters





<u>Figure 3</u>
$N_l = 200$
Time steps increase as Relative Error decreases and H Factor increases

<u>Figure 4</u>
Relative Error = 0.02
Time steps increase as H Factor increases, but seem independent of NI

Methods: Bond Pricing (Stochastic Interest Rates)

Equation (1)

$$\begin{cases} \mathrm{d}r_t = \alpha(\theta_t - r_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^1, \\ \mathrm{d}\theta_t = \beta(\phi - \theta_t)\,\mathrm{d}t + \eta\,\mathrm{d}W_t^2, \end{cases}$$

Equation (2)

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,\mathrm{d}u} P_T(T) \middle| \mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,\mathrm{d}u} \middle| \mathcal{F}_t\right]$$

Equation (3)

$$\mathrm{d}X_t = \mu(t, X_t)\,\mathrm{d}t + \sigma(t, X_t)\,\mathrm{d}B_t$$

$$X_t = (r_t, \theta_t)$$

$$\mu(t, x, y) = \begin{bmatrix} \alpha(y - x) \\ \beta(\phi - y) \end{bmatrix}$$

$$\sigma(t, x, y) = \begin{bmatrix} \sigma & 0 \\ 0 & \eta \end{bmatrix}$$

Feynman-Kac Theorem

Equation (4)

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u\,\mathrm{d}u} \middle| \mathcal{F}_t\right] = u(t, r_t, \theta_t),$$

Methods: Bond Pricing (Stochastic Interest Rates)

$$P_t(T) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T r_u \,\mathrm{d}u} \middle| \mathcal{F}_t\right] = u(t, r_t, \theta_t)$$

PDE

$$\begin{cases} \dfrac{\partial u}{\partial t} + \alpha(y-x)\dfrac{\partial u}{\partial x} + \beta(\phi - y)\dfrac{\partial u}{\partial y} + \dfrac{1}{2}\sigma^2\dfrac{\partial^2 u}{\partial x^2} + \dfrac{1}{2}\eta^2\dfrac{\partial^2 u}{\partial y^2} = xu \\ u(T,x,y) = 1. \end{cases}$$

SDE

$$P_t(T) = e^{A_t(T) - B_t(T) r_t - C_t(T)\theta_t}$$

A_t(T), B_t(T), C_t(T) are known functions

Euler Maruyama Discretization
MLMC Path Sampling

Neural Network Model

After Investigation of Bond Pricing:

- Pricing Bond Options
- Pricing Swaptions

Conclusion and Next Steps

- Comparing MLMC and MC
- Exploring MLMC Hyperparameters
- Bond Options (Stochastic Interest Rates)

Conclusion

Key Results

- MLMC demonstrates significant time complexity improvement over MC with an improvement of 5.1-7.2x less time steps and 1.4-1.6x less overall training time
- Number of required time steps increases as Relative Error decreases and H-Factor increases
- MLMC and DNN's performance with Stochastic Interest Rate can be investigated through applying Feynman-Kac to Zero Coupon Bonds

Next Steps

- Explore MLMC and DNN's with complex Stochastic Interest Rates PDEs at high dimensions for Zero Coupon Bonds, Bond Options, and Swaptions
- Explore the optimization of MLMC hyperparameters for minimizing number of time steps required

Thank you.

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